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LETTER TO THE EDITOR

On the integrability of nonlinear discrete systems

Jesus S Dehesa and A Lallena
Departamento de Física Nuclear, Facultad de Ciencias, Universidad de Granada, Spain

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Abstract. The matrix formulation of Flaschka is used to study the integrability of nonlinear N-particle systems of exponential type (e.g. Toda lattices, electric circuits of ladder type, Volterra systems). The asymptotic, i.e. $N \to \infty$, behaviour of the normalised eigenvalue moments $\{\mu_r'^{(N)}; r = 0, 1, \ldots, N\}$ of the N-dimensional L matrices, which are constants of motion of the system, is investigated. Compact expressions of these quantities in terms of the asymptotic values of the dynamical variables (P_n, Q_n) of the Toda lattice are analytically obtained in a simple way. The applicability of these expressions is illustrated in the case $(P_n \to 0, Q_n - Q_{n-1} \to 0)$, which encompasses many of the motions of the Toda lattice considered in the literature.

The Toda lattice (Toda 1970) is a one-dimensional lattice of N particles with nearest-neighbour interactions of exponential type. It can be imagined as a system of N particles connected by nonlinear springs with an exponential restoring force (Toda 1975, 1976). This nonlinear discrete Hamiltonian system was suggested numerically (Ford *et al* 1973) and proved analytically (Hènon 1974, Flaschka 1974a, b, Manakov 1975) to be completely integrable, i.e. a system whose Hamiltonian can be reduced to an obviously solvable form by a canonical transformation analytic in the position, Q_n, and momentum, P_n, variables. This feature makes the Toda lattice a physically interesting nonlinear system which can be analytically solved exactly, e.g. it is intimately connected with other relevant Hamiltonian N-body systems (Gibbon 1978), and its equation of motion generates in the continuum limit integrable nonlinear evolution equations such as Korteweg–de Vries, Boussinesq and others (Toda 1975, 1976, 1978, Gibbon 1978). Moreover, the existence of solitons in a nonlinear system seems to be connected (Flaschka 1975) with the complete integrability of its equation of motion.

The Hamiltonian of the Toda lattice is given (Toda 1975) by

$$H = \sum_{n=1}^{N} \left(\tfrac{1}{2}P_n^2 + e^{-(Q_n - Q_{n-1})}\right) \tag{1}$$

and the corresponding equations of motion can be written as follows:

$$dQ_n/dt = P_n, \qquad dP_n/dt = e^{-(Q_n - Q_{n-1})} - e^{-(Q_{n+1} - Q_n)}.$$

These equations can be expressed (Flaschka 1974a, b, Manakov 1975) in matrix form as

$$dL/dt = BL - LB \tag{2}$$

where the N-dimensional matrices B and L are such that their only non-vanishing

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elements are

$$\begin{cases} B_{i,i+1} = b_i \\ B_{i+1,i} = -b_i \end{cases} \quad \text{and} \quad \begin{cases} L_{ii} = a_i \\ L_{i+1,i} = L_{i,i+1} = b_i \end{cases} \tag{2a}$$

respectively. Here

$$a_n = -\tfrac{1}{2}P_{n-1}, \qquad b_n = \tfrac{1}{2}\exp[-\tfrac{1}{2}(Q_n - Q_{n-1})], \tag{3}$$

the boundary conditions being (Moser 1975a, Kotera and Yamazaki 1977) $b_0 = 0$ and $b_N = 0$, which corresponds to $Q_0 = -\infty$ and $Q_{N+1} = +\infty$.

These eigenvalues $\{\lambda_i; i = 1, \ldots, N\}$ of the matrix L are independent of time (Flaschka 1974a, b), which means that they are integrals of the motion or that the motion in the lattice is characterised as an isospectral deformation. Since the expressions of λ_i in terms of the canonical variables $\{P_n, Q_n; n = 1, \ldots, N\}$ are extremely complicated, new families of integrals of motion were introduced. These families, which are symmetric functions of the λ_i, show two important features: they are also important conceptually and can be expressed as relatively simple functions of the canonical variables.

One of these families of N constants of motion is that formed by the so-called elementary symmetric functions (Bellman 1970) of the eigenvalues of L, i.e. $I_1 = \Sigma_{i=1}^{N} \lambda_i$, $I_2 = \Sigma_{i\neq j} \lambda_i\lambda_j, \ldots, I_N = \lambda_1\lambda_2\lambda_3 \ldots \lambda_N$, which are the coefficients in the expansion of the characteristic polynomial of L in powers of λ, namely

$$\det(\lambda 1 - L) = \sum_{m=0}^{N} I_m \lambda^{N-m}.$$

A second family of integrals of motion is $\{J_m; m = 1, \ldots, N\}$, where J_m is defined by

$$J_m[(-2)^m/m]\,\mathrm{Tr}\,L^m - 2N(-1)^N\delta_{m,N} \tag{4}$$

for $m = 1, 2, \ldots, N$ and $\delta_{m,N}$ is the Kronecker delta.

Both sets of integrals were introduced (Hènon 1974) for a periodic Toda lattice, i.e. b_N is not zero in general. This author also found closed expressions for the integrals in terms of the canonical variables, i.e. expressions of the form $I_m = F_m(P_1, P_2, \ldots, P_N, Q_1, \ldots, Q_N)$, $J_m = G_m(P_1, \ldots, Q_N)$ for $m = 1, 2, \ldots, N$. Contrary to the other set, the integrals J_m can usually be extended to an infinite lattice in a simple way. However, the asymptotic (i.e. $N \to \infty$) behaviour of these integrals has not yet been investigated. Further, a detailed study of the asymptotic density of eigenvalues of the matrix L in terms of the asymptotic behaviour of the canonical variables (P_n, Q_n) is still missing. It is the purpose of this Letter to contribute to this study what seems to us to be necessary to establish firmly a theory of the Toda lattice (Flaschka 1975, Moser 1975b).

More precisely, we will investigate the collective spectral properties of the matrix L given in ($2a$) when $N \to \infty$. These average or global eigenvalue properties can be completely characterised by means of the moments $\{\mu_m'^{(\infty)}; m = 0, 1, \ldots\}$ of the asymptotic eigenvalue density defined as follows:

$$\mu_m'^{(\infty)} = \lim_{N\to\infty} \mu_m'^{(N)} = \lim_{N\to\infty} \frac{1}{N} \sum_{i=1}^{N} \lambda_i^m \tag{5}$$

where the moments $\mu_m'^{(N)}$ of the discrete eigenvalue density of the N-dimensional matrix L are connected with the integrals J_m by

$$J_m/N = [(-2)^m/m]\mu_m'^{(N)}. \tag{6}$$

Notice that the moments are normalised so that $\mu_0'^{(\infty)} = \mu_0'^{(N)} = 1$.

Our main result is the following theorem, which gives the asymptotic density of eigenvalues of the matrix L corresponding to a Toda lattice with fixed-end boundary conditions by means of its moments $\mu_m'^{(\infty)}$ and the asymptotic behaviour of the integrals of motion J_m in terms of the values of the canonical variables (P_n, Q_n) when $n \to \infty$.

Theorem. If

$$\lim_{n\to\infty} P_n = p \in \mathbb{R} \qquad \text{and} \qquad \lim_{n\to\infty} (Q_n - Q_{n-1}) = q \in \mathbb{R} \tag{7}$$

then

$$\mu_m'^{(\infty)} = (-p/2)^m \sum_{j=0}^{[m/2]} e^{-qj} p^{-2j} \binom{2j}{j}\binom{m}{2j} \tag{8}$$

for $m = 0, 1, 2, \ldots$, and, taking into account (6), it turns out that

$$\lim_{N\to\infty} \frac{J_m}{N} = \frac{p^m}{m} \sum_{j=0}^{[m/2]} e^{-qj} p^{-2j} \binom{2j}{j}\binom{m}{2j}, \tag{9}$$

where $[m/2]$ is equal to $m/2$ or $(m-1)/2$ when m is even or odd respectively.

Proof. To prove this theorem, we use the following result (Dehesa 1978) concerning Jacobi matrices of the L form given in $(2a)$: if

$$\lim_{n\to\infty} = a \in \mathbb{R} \qquad \text{and} \qquad \lim_{n\to\infty} b_n = b/2 \geqslant 0 \tag{10}$$

then

$$\mu_m'^{(\infty)} = \sum_{j=0}^{[m/2]} b^{2j} a^{m-2j} 2^{-2j} \binom{2j}{j}\binom{m}{2j}. \tag{11}$$

Replacing in (10) the values of a_n and b_n of a Toda lattice given by (3), one obtains the conditions (7) in a straightforward manner. It turns out that

$$a = -\tfrac{1}{2}p \qquad \text{and} \qquad b = e^{-q/2}.$$

With these two values the expression (11) easily reduces to (8).

Although this theorem has been stated and proved for the dynamical variables (P_n, Q_n) which characterise a Toda lattice, it can be extended in a straightforward manner to other similar nonlinear systems, such as the ladder-type electric circuits (see e.g. Hirota and Suzuki 1970, 1973, Hirota and Satsuma 1976) in network theory, and the Volterra systems which play an important role in plasma physics (see Zakharov *et al* 1974, Manakov 1975). To do that, it suffices to consider the well-known simple relations between the canonical variables of these systems and the corresponding variables P_n, Q_n of the Toda lattice (see e.g. Wadati 1976). In fact, the Toda-like potential is the only possible one to make the one-dimensional N-particle system integrable, given the assumption of the nearest-neighbour interaction (Sawada and Kotera 1976).

To illustrate the applicability of our theorem, we shall consider those motions of the Toda lattice corresponding to the collection of sequences P_n, Q_n satisfying the two following conditions (Flaschka and McLaughlin 1979):

$$\lim_{n\to\infty} P_n = 0 \qquad \text{and} \qquad \lim_{n\to\infty} (Q_n - Q_{n-1}) = 0. \tag{12}$$

The comparison of these expressions with (7) gives $p = 0$ and $q = 0$. Therefore the theorem gives for the asymptotic eigenvalue moments $\mu_m'^{(\infty)}$ the values

$$\mu_m'^{(\infty)} = \begin{cases} 1 & \text{for } m = 0 \\ 0 & \text{for } m \geqslant 1. \end{cases} \tag{13}$$

The simplest case satisfying the conditions (12) is when the Toda lattice has no motion, since then $P_n = 0$ and Q_n is a real constant. Besides, an interesting case which also satisfies (12) is the soliton (Toda 1975) defined by the canonical variables P_n and Q_n,

$$P_n = \beta(\tanh \theta_n - \tanh \theta_{n-1}), \qquad \exp[-(Q_n - Q_{n-1})] = 1 + \beta^2(1 - \tanh^2 \theta_{n-1}),$$

with $\theta_n = \alpha n - \beta t$ and $\beta = \sinh \alpha$. Indeed, one can easily prove that conditions (12) are also satisfied. Therefore the values (13) for $\mu_m'^{(\infty)}$ are also valid in this case, which may be intuitively understood from the concept of the soliton.

Let us finally make some remarks. To establish firmly a theory of the Toda lattice, a detailed study of the (asymptotic) spectrum of eigenvalues of the matrix L is needed. Especially important from a physical point of view is the investigation of the (asymptotic) collective properties of this spectrum, e.g. the eigenvalue density distribution, since it includes the relevant integrals of motion of the lattice; see equation (6). In this Letter, we have found the moments $\mu_m'^{(\infty)}$ of the asymptotic eigenvalue density of the matrix L corresponding to a discrete (i.e. N-particle) Toda lattice with fixed ends directly in terms of the asymptotic behaviour of the canonical variables which characterise the lattice. As a result, the asymptotic behaviour of the integrals of motion first introduced by Hènon is easily obtained. The application of this theorem is illustrated in the calculation of $\mu_m'^{(\infty)}$ for a case which encompasses many of the motions of the Toda lattice.

Since the eigenvalues of the matrix L represent (Flaschka 1974a, b, 1975, Flaschka and McLaughlin 1979) in certain circumstances the normal modes of oscillation for the Toda lattice, it is physically important to remark that the knowledge of all ($\mu_m'^{(\infty)}$) $\mu_m'^{(N)}$ determines the (asymptotic) density of nonlinear normal modes of the (infinite) Toda lattice (Jackson 1978). To end up, let us point out that it would be desirable to extend the results of this Letter in the following two directions: firstly to Toda lattices with fixed-end boundary conditions which are not subject to the restrictions (7), and secondly to periodic Toda lattices. Work in this connection is being done.

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